Mind in Mathematics:
Essays on Mathematical Cognition and Mathematical Method

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Rhythm as an Integral Part of Mathematical Thinking

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The topic of this chapter is rhythm. The argument that I will be defending is that rhythm is an integral part of mathematical thinking. The argument rests on the idea that mathematical thinking is something akin to temporal art, like drama, poetry, and dance—mathematical thinking happens *in time*. The argument goes even further. Mathematical thinking not only happens in time but its most striking feature is *movement*.

This conception of mathematical thinking is at odds with traditional conceptions that portray thinking as something happening in a kind of mental repository containing ideas:

> We cannot speak of ideas as being in the mind in the sense in which furniture is in the room; an idea is an active thing constituted by its activity, coloring other ideas and in process of internal modification; it must be conceived as a form of energy (Demos 1933: 273)

Thinking, as I conceptualize it here, is rather "an effluence of creativity; it has come about and it may give place to some other manifestation of life" (Demos 1933: 273). To assert that mathematical thinking is movement brings us close to the topic of rhythm. Rhythm has a convoluted etymology. It was defined in the 16th century as something related to the manner in which the accentuation of syllables affects the oral reception of language; it might not be a surprise then, that thus conceived, rhythm remained entangled for many years with its measuring form—metrics.

Referring to the linguistic tradition, where rhythm has been investigated through prosody, that is, the linguistic patterns of stress and intonation, Richard Cureton (2004: 113) notes that "The prosodic tradition has always been primarily interested in the voice, how it moves rhythmically from syllable to syllable, stress to stress." Rhythm, however, went beyond the linguistic tradition and started being applied to other domains like music, and natural and social phenomena. In its general sense, the concept of rhythm tries to characterize the reappearance of something at regular intervals and attempts to capture the idea of regularity, alternation, or something oscillating between symmetry and asymmetry. Here, rhythm appears as a complex of conflicting "components," each one exploring and expressing our experience of the world in a different manner. Each one of these components creates a different sort of subjective time:

> Meter creates *cyclical time*, which is associated with sensation, perception, and physical ecstasy. Rhythmic grouping creates *centroidal time*, which is associated with the centered self and emotional expression. Prolongation creates *linear time*, which is associated with volition and action. And theme creates *relative time*, which is associated with thought, imagination, and memory (Cureton 2004: 114)

Although Cureton's components were conceived as a means to offer a more encompassing conception of prosody, they apply to rhythm in general, where "rhythm is not just the regular movement of the voice but the concerted and conflicted movements of our major psychological faculties—sensation, emotion, volition, and thought; body, soul, will, and mind" (Cureton 2004: 114). This is why rhythm is at the basis of creativity, meaning production, and learning. It is not be by chance, indeed, that rhythm has been considered a central element of language acquisition. Some studies suggest that "newborns use prosodic and, more specifically, rhythmic information to classify utterances into broad language classes defined according to global rhythmic properties" (Nazzi, Bertoncini, and Mehler 1998: 756).

This chapter is divided into two parts. In the first part I suggest a distinction between mathematical thought and mathematical thinking that draws on the Aristotelian distinction between *potentiality* and *actuality*. In the second part I discuss a Grade 9 classroom example to illustrate the ideas put forward in the chapter.

Thought and thinking

In the previous section I suggested that thinking (think-*ing*) is movement. Now, since "There is no movement which is not the movement of something" (Adorno 2001: 57), we need to specify that which is put into motion when thinking occurs. To advance the answer that I elaborate below, this something is *thought*.

The conception of thought and thinking that I put forward here can be better understood in reference to two related although different conceptual categories introduced by Aristotle: *potentiality* and *actuality*. Potentiality (*δύναμις, dunamis*) designates the source of motion and, as its name suggests, is a *dynamic* principle: potentiality is to have a definite capacity *for doing something*. It is synonymous with "capacity" or "ability" or "power." Living things and artifacts have potentiality. A musical instrument, for example, has the capacity to produce sounds. A fish has the capacity to move in the water. Aristotle contrasted this potentiality to *actuality* (*ἐνέργεια, energia*), which is "being-at-work" (Sachs 2015: 3)—something in motion occurring in front of us, as in our examples, the actual sound produced by the musical instrument or the path travelled by the fish to go from one point to another. Potentiality is hence a pure possibility; something indefinite, without shape: that which is merely potential and that, through movement, becomes *materialized* or *actualized*—the sound emitted by the instrument or the path followed by the fish.

The relationship between thought and thinking that I would like to convey here is the relationship between potentiality and actuality. More precisely, I suggest that thinking is the *materialization* or *actualization* of thought. Thought, then, is pure possibility—the possibility of thinking about a certain state of affairs in a certain manner; for instance, the possibility of thinking about numbers in a definite way (a specific way of thinking about quantities as such or such). Thinking is the materialization or actualization of such a possibility—the actual thinking about quantities in such or such a way. It is the actualization or real occurrence of what, so far, was pure possibility.

Now, potentiality in living things and artifacts may be present as a natural feature or as an acquired one. The fish is biologically and instinctually equipped to move in the water. Other capacities or abilities, however, as Aristotle notes in Book Theta 5 of *Metaphysics* (1048a), are the result of education (Aristotle 1998: 263). This is the case of mathematical thinking. Thinking mathematically is not something that unfolds naturally like the fish's motion in a river. Although we may have a basic and elementary raw range of orienting biological reactions that we share with some primates and other species and that goes back to prehistoric hunters (de Almeida 2013), to think algebraically about sequences or equations, or to think about shapes and forms as we do today in Cartesian or projective analytic geometry, requires the formation or appearance of potentialities that cannot merely emerge as the result of natural evolution. They can only emerge as the result of cultural-historical development. This is why we should be better off understanding potentiality and actuality as two interrelated entities that evolve culturally. The task, of course, is not easy. Western rationalist and empiricist epistemologies have traditionally adopted the view according to which actuality (thinking in our case) is unrelated to potentiality, that is, to its conditions of possibility. Plato's epistemology is a good example. In Plato's account, ideas are conceived of as self-subsistent, as absolute entities that exist in themselves and by themselves, that is, as entities independent of the conditions of their realization. The same is true of Hume's empiricist view of ideas. Even if Aristotle did not bring the problem of thinking and its conditions to its last consequences (for reasons that I shall mention below), there is room—*in principle*—in the Aristotelian epistemological framework for understanding thinking as related to its conditions of possibility (potentiality).

To make my point clear, let us go back to Aristotle's terminology. Potentiality corresponds to what Aristotle calls pure *matter*, while actuality corresponds to *form*. Since, within this view, matter can be realized or manifested in various ways, the problem appeared as the problem of the "One in the Many" (matter and its many actualizations or instantiations). In Kant's (2003) work, the problem reoccurs as the problem of the synthesis of unity in diversity, that is, the subsuming of particular instances of something into their corresponding concept. Following the Platonist tradition, Kant considers the unity (or the concept) as something a priori, independent of the individuals' experiences. Now, in Aristotle's work, matter is not separated from form. His *Metaphysics* turns around the claim that form cannot be dissociated from the matter it actualizes. As Adorno (2001: 34) summarizes it, there is in Aristotle's work a reciprocal interrelatedness of matter and form expressed in the thesis that "the One exists only as the unity of the many, and that the Many exists only as a manifold of units." The Circle exists only as the unity of many sensible circles. The diversity of sensible circles exists as the manifold of the Circle. The central problem of Aristotle's account becomes the problem of the realization of matter in form. Neither potentiality (of which I am suggesting that thought is an example), nor actuality (of which I am suggesting that thinking is an example) is something constituted in itself and by itself. Potentiality, for instance, does not refer to something *transcendental*, but to something *immanent* that manifests itself in its actualization in a *res singularis*. This is why thought is immanent in thinking and thinking can only become into existence as what results from the motion of thought. It is in this sense that thought is not being, but rather something potential that is becoming through a movement that implies its own incompleteness and the continued presence of both potentiality and actuality (Ross 2005).

The mediation of thought and thinking

If it is true that Aristotle asserts the interrelatedness of matter and form, this interrelation—which, in our case, refers to the interrelation of thought and thinking— is, however, conceived of as *unmediated*. The relationship between matter and form is a simple relation of connection. Adorno suggests that the Aristotelian relationship of matter and form can be pictured as "an amalgam" and not as a "chemical compound, in which the two apparently antithetical elements [would be] so fused that one cannot exist without the other" (Adorno 2001: 56). In the Hegelian dialectical account of thought and thinking that I would like to sketch here, thought, in the movement that transforms it from pure possibility into the form of energy that thinking is, does not simply vanish but remains embedded (or *sublated,* to use Hegel's term) *in* thinking.

Behind this idea is the more general idea of life as a continuously evolving whole system that Hegel outlines in the famous Preface to his *Phenomenology of Spirit*. Hegel considers the example of a bud:

> The bud disappears in the bursting-forth of the blossom, and one might say that the former is refuted by the latter; similarly, when the fruit appears, the blossom is shown up in its turn as a false manifestation of the plant, and the fruit now emerges as the truth of it instead. These forms are not just distinguished from one another, they also supplant one another as mutually incompatible. Yet at the same time their fluid nature makes them moments of an organic unity in which they not only do not conflict, but in which each is as necessary as the other; and this mutual necessity alone constitutes the life of the whole (Hegel 1977: 2).

The "dialectical moment" (Hegel 1991: 128) is to be found in the sublation of the bud and its transformation into its opposite (the blossom, that is, that which the bud is not, is something else that in its otherness "refutes" the bud). The dialectical moment is also to be found in the "negation" of the blossom and its sublation into the fruit. "It is of the highest importance," Hegel notes, "to interpret the dialectical [moment] properly, and to [re]cognise it. It is in general the principle of all motion, of all life" (Hegel 1991: 128).

The aforementioned oppositions (or "contradictions," as Hegel often calls them) that lie at the heart of the dialectical moment are not logical flaws but part of the process of life:

> By the presence of contradictions. Hegel plainly means the presence of opposed, antithetical tendencies, tendencies which work in contrary directions which each aim at dominating the whole field and worsting their opponents, but which each also require these opponents in order to be what they are (Findlay 1958: 77).

Likewise, I would like to suggest, through a series of dialectical moments, that thought develops into thinking and thinking into new thinking. In the course of this process, thinking synthesizes itself, incorporating or sublating previous formations—like the bud into the blossom and the blossom into the fruit—and creates new thought for more thinking to occur. In the course of this transformation, thought is sublated into thinking and thinking appears not as something *immediate*, but as the result of a *mediation*.

It is indeed in this sense that Hegel uses for the first time the term *mediation* in the *Encyclopaedia* (Hegel 1991). When the empirical world is the object of thinking, the empirical world is *transformed* (or "elevated," as Hegel says) into something else, into a concept: "this elevation is a *passage* and *mediation*, it is also the *sublating* of the *passage* and the *mediation* (Hegel 1991: 96; emphases in the original). Hence, mediation appears as a *process*—the process that transforms something into something else. Mediation "is to take something as a beginning and to go onward to a second thing; so that the existence of this second thing depends on our having reached it from something else contradistinguished from it" (Hegel, 2009, p. 135). Rather than a mere immediate thing, this second thing carries now, in a refracted and sedimented way, not only the history of its mediation, but also, in a sublated manner, the traces of its conditions of possibility (its beginning). Following Adorno's chemical metaphor, we can say that the second thing is not a second thing *qua* second thing, but it is rather a chemical compound where the first and second things mediate each other.

In the terminology of the previous sections, we can formulate mediation as that which puts matter into motion and transforms it into a form. It is indeed in this triadic schema that mediation appears and reappears in Hegel's *Logic*. It is through the action of the mediating term that matter (pure potentiality) is filled with particular determinations and makes it cognizable as a form. In the particular case of thought, that which puts it into motion and fills it with content and relations with other contents in order to materialize or actualize itself in thinking, this mediating term is *human sensuous practical activity*. Figure 1 provides a graphical meaning of these ideas.

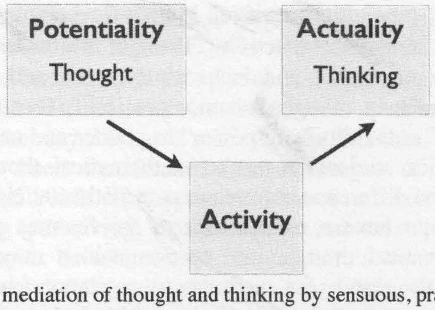

Figure 1. The mediation of thought and thinking by sensuous, practical activity

We are now ready to return to rhythm and to illustrate how, in the mediation of thought, rhythm constitutes a central feature of mathematical thinking.

Rhythm in mathematical thinking

As mentioned above, the question of what puts matter into motion is one of the crucial questions of Aristotle's *Metaphysics*. For Aristotle, the "Prime Mover," that is, the mover that does not need a mover to set things into motion, is the incorporeal spirit. "The ultimate ground of all movement, therefore—to state the matter in Aristotelian terms—is the divinity itself as pure and perfect mind or spirit (*Geist*)" (Adorno 2001: 91). Adorno interprets Aristotle's solution within the context of a philosophical perspective that grew up in the midst of an agonizing democracy and the ensuing evanescence of *praxis* in the *polis,* where the separation of intellectual and manual labor has been achieved, culminating in the glorification of pure theory.

> Political praxis, as it had been carried on in accordance with traditional Greek—that is, Athenian or Attic—democracy, was no longer possible. And out of this necessity, this deprivation, the metaphysicizing of theory, which was taken to be the principle of the divinity itself, made not only a virtue, but the highest virtue. (Adorno 2001: 92)

Max Horkheimer adds that "Movement as such, detached from its social context and its human aim, becomes the mere appearance of movement" (quoted in Adorno 2001: 172).

In the previous section I have suggested that it is joint, practical, sensuous material activity that sets thought into motion. Before motion, thought is pure possibility. When the students cross the threshold of the school for the first time, mathematical thought is for them pure open possibility—for example, the possibility of thinking about numbers in a definite (e.g., arithmetic or algebraic) way. But, as briefly mentioned in the previous section, thought as possibility is not something eternal, static, or independent of all human experience (as in Kant's concept of things-in-themselves or as in Plato's forms). In fact, thought results from, and is produced through, human social activity. Thought is a *culturally codified synthesis* of the ways in which people reflect, act, and deal with the world and with each other. Through the codification of the synthesis, thought becomes possibility for thinking. Thought presents itself as a pure "capacity" or "ability" or "power" to ponder and achieve things. Both expert mathematicians and novice students resort to mathematical thought when they tackle a mathematical problem. The difference between expert mathematicians and novice students is that, in general, the former have a clearer idea of the features and particularities of the activity that sets mathematical thought into motion, which may lead us to believe that thinking is unmediated—the result of a spontaneous cogitation (the spontaneous activity of the spirit). This apparent immediacy is, indeed, the result of a long education. Hegel says:

> Like anyone who has been instructed in a science, a mathematician has solutions at his fingertips that were arrived at by a very complicated analysis; every educated human being has a host of general points of view and principles immediately present in his knowing, which have only emerged from his meditation on many things, and from the life experience of many years (Hegel 1991: 115)

The apparent immediacy of the mathematician thinking does not mean that thinking has not been mediated. The mediation appears sublated into thinking itself.

Let us consider the following sequence:

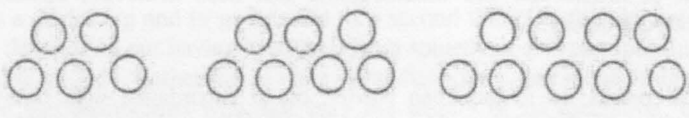

Figure 2. The three first terms of a sequence

How many circles are there in Term 10 and in Term 100? We have informally asked some fellow mathematicians these questions and most of the answers can be grouped into three main categories. Mathematicians tend to see the terms of the sequence as made up of two rows. Then they scan the rows for functional clues between the number of the term and the number of circles on one row and the other. In general, mathematicians quickly realize that there is one more circle on the top row than the number of the term, and that there are two more circles on the bottom row than the number of the term. They conclude that the general formula is $y = n+1 + n +2$, that is $y = 2n + 3$. Then, they apply the formula to Terms 10 and 100. In other cases, mathematicians perceive the circles as placed in couples through diagonals. Hence, Term 1 has 2 × 2 + 1 circles. They generalize to Term n, which has 2(n+1) + 1, or 2n + 3. The third kind of answer consists of noticing the recursive relationship $T_{n+1} = T_n + 2$ (which allows one to identify the sequence as an arithmetical one) and to transform the repeated addition into a multiplication. The point is that perception and the ensuing calculations go so fast that mathematicians end up thinking that it is natural to perceive sequences in those ways. What is missed is the fact that, to arrive at these ways of perceiving, the eye has been transformed from an organ of naïve perception into a *theoretical organ* (Radford 2010). This perceptual transformation is the lengthy process of education, where thinking (perceiving, calculating, etc.) about sequences becomes a kind of *second nature* that seems to be unmediated. In Aristotle's terminology, *matter* seems to merely vanish and *form* seems to be self-sufficient. The conditions of possibility of thinking (i.e., thought) seem to merge into thinking with the result that we have the impression that thinking in this (mathematical) way is ubiquitous and natural. Our research with young (seven-to-eight-year-old students) shows the opposite. Indeed, our research shows very clearly that young children tend to see the terms of the sequence as a bunch of circles and focus on the numerosity of the terms, without trying to capitalize on the spatial arrangement of the circles (Radford 2012, 2014). The perceptual activity that lies at the bottom of algebraic thinking is, for them, pure possibility. It is a possibility whose phylogenetic formation goes back to the Mesopotamians, the Pythagoreans, and the neo-Platonists of Mediterranean antiquity and that has been refined and developed over the course of cultural history.

The educational challenge is hence the following. In order for the students to encounter the culturally developed contemporary forms of mathematical thinking, suitable classroom *activities* need to be conceived. What is meant by the suitability of these activities (e.g., their content, conceptual structure, etc.) has been at the heart of many discussions in education and certainly there is no consensus about it. Within the theoretical framework sketched in the previous sections, these activities acquire an extremely important role as they are conceived of as *mediations*: it is through activity that pure possibility is actualized in something concrete, namely specific forms of thinking mathematically.

It is in the course of the meditational role of the classroom activities that mathematical thinking emerges against the background of culturally evolved thought. Its emergence is not a mere intellectual experience, but a fully corporeal, embodied, and sensuous phenomenon where rhythm in its various modalities appears. I devote the rest of this chapter to illustrate this point. My discussion turns around a Grade 9 lesson where 14-15-year-old students were dealing with the sequence shown in Figure 2. The goal of the lesson was to offer the students an opportunity to think about figural sequences in an algebraic manner. To do so, the students were invited to work in small groups of three. In the first part of the lesson, the students were asked to continue the sequence, drawing Term 4 and Term 5 and then to find out the number of circles in Figure 10 and Figure 100. In the second part, the students were asked to write a message explaining how to calculate the number of circles in any figure (*figure quelconque*, in French) and, in the third part, to write an algebraic formula. I focus here on a three-student small group formed by Jay, Mimi, and Rita.

Jay had the activity sheet in front of him. Looking at the terms on the activity sheet, the students started by counting the number of circles and realized that they increase by two each time. This is the third category of the mathematicians' strategies to which we referred above. However, in drawing Term 4, they switched to another strategy and started perceiving the terms as made up of two rows, which corresponds to the first category of the mathematicians' strategies alluded to above:

1. Rita: You have five here . . . (pointing to Term 3 on the sheet).
2. Mimi: So, yeah, you have five on top (she points to the sheet, placing her hand in a horizontal position in the space in which Jay is beginning to draw Term 4; see Figure 3) and six on the [bottom] (she points again to the sheet, placing her hand a bit lower).

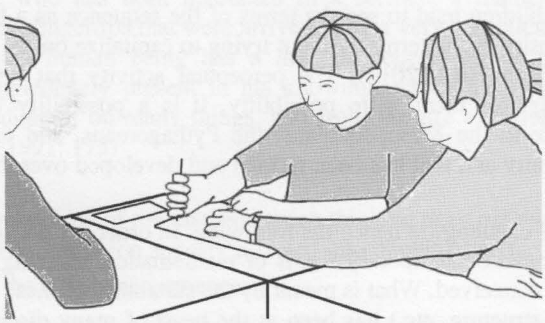

Figure 3. Rita to the left, Jay in the middle, and Mimi to the right. The sketch shows Mimi's first gesture on line 2

The conceptual content of Rita's turn in line 1 is not merely informative. She is not merely saying that Term 4 contains a row of five circles. In fact, through an indexical gesture, she suggests a qualitative and quantitative manner to perceive the terms of the sequence. Although she does point to a specific part of Term 3, which is given on the sheet, she is referring to Term 4. In the sense-making process, Term 3 stands for the not-yet-visible Term 4. This is an example of a process of *perceptual semiosis*: a process in which perception is continuously refined through (embodied and mathematical) signs.

In line 2, drawing on Rita's idea, Mimi describes Term 4 through the spatial deictics "top" and "bottom," shifting from blunt counting to a proper and efficient scheme of counting. She accompanies her words by two rhythmic corresponding deictic gestures, which allow her to depict the spatial position of the rows in an iconic way. This two-row-based counting scheme is the first step in the process through which thought as pure possibility is endowed with concrete determinations. From something fuzzy and general, thought (as pure cultural possibility) is set into motion and becomes shaped, refined, and specified. It is actualized in a form of algebraic thinking. Of course this algebraic form of thinking is not yet expressed in a full alphanumeric symbolism. It is already algebraic nonetheless. It is an embodied form of thinking expressed through collective gestures, perception, words, and signs.

Following a similar path, the students continued their task and drew Term 5 of the sequence, doing it in a more direct way. Then, they turned to Terms 10 and 100:

3. Mimi:	Term 10 would have…
4. Jay:	10 there would be like…
5. Mimi:	There would be eleven (she makes a quick gesture that points to the air) and there would be ten, right?
6. Jay:	Eleven (making a gesture over the sheet; see Figure 4, left) and twelve (making a second gesture over the sheet; see Figure 4, right).
7. Mimi:	Eleven and twelve. So it would make twenty-three, yeah.
8. Jay:	[Term] 100 would have one-hundred-and-one and one-hundred-and-two (making a couple of gestures similar to those in line 6).

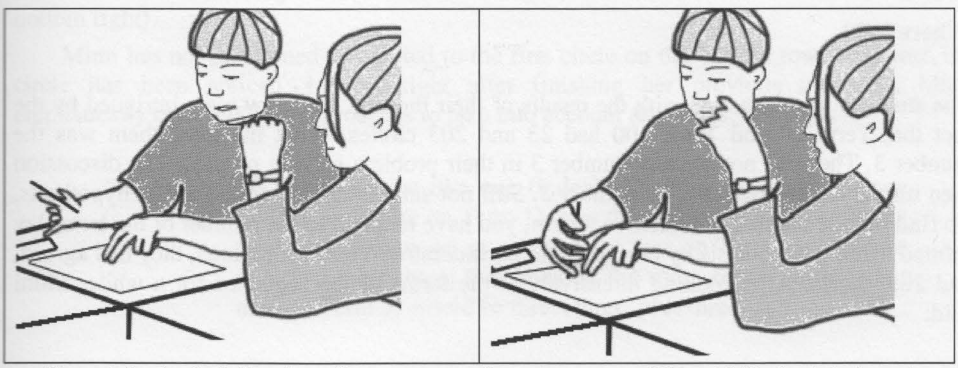

Figure 4. Jay (to the left) makes a first gesture to refer to the top row of Term 10. To the right, he makes a second gesture to refer to the bottom row of Term 10

Terms 10 and 100 are not perceptually available. Jay makes gestures in the air in a rhythmic way as if pointing to the rows of the invisible terms.

I would like to suggest that rhythm is a powerful element in the unfolding of the students' thinking and in its organization. Jay's rhythmic gestures in line 6 involve "meter," which, as Cureton (2004: 114) suggests, is associated with sensation and perception. Meter helps Jay to imagine Term 10 as made up of two rows containing specific numbers implicitly related to the number of the term. But the rhythmic display of Jay's gestures and words evokes also the possibility to apply the same counting schema to other terms of the sequence. In Cureton's terminology, this is related to *theme* and *prolongation*. That what has been noticed, grasped, and expressed is not true of Term 10 only; it is true of the other still non-spoken terms. Meter hence opens a possibility for new things to be imagined (theme) and expressed (prolongation). New terms can now be investigated, prolonging the actual experience beyond its current limits. And indeed, as shown in line 8, Jay makes a second group of similar gestures to refer to Term 100. Considered together, the two couples of rhythmic gestures (lines 6 and 8) constitute a *rhythmic grouping*, that is, a new entity associated, as Cureton suggests, with the centered self and emotional expression. Indeed, Jay is not merely saying things, like *Siri* in an iPhone. Jay and his group-mates are emotionally involved. Through fully emotional utterances, and Jay's gestures and their positional location in space, a centered self is defined vis-à-vis the imagined terms and the other students.

Rather than ephemeral byproducts of communication and problem-solving heuristics, these elements of rhythm—meter, rhythmic grouping, theme, and prolongation—are essential components of the flow of thinking. They are central features of the mediation of thought and the manner in which it becomes actualized in the students' reflections and actions. They are part of the *materiality* of thinking.

Where is 3?

The students were satisfied with the results of their inquiry. Yet, they were intrigued by the fact that Term 10 and Term 100 had 23 and 203 circles. What intrigued them was the number 3. They did not see any number 3 in their problem-solving process. The discussion then turned to making sense of number 3. Still not sure, Mimi suggested a first hypothesis. To find out the number of circles in a term, you have to add 3 to the number of the term. Jay refuted Mimi's hypothesis by bringing into consideration Term 100, which, they had agreed, had 203 circles. After looking intensively at the terms of the sequence for a while, Mimi said:

9. Mimi: You know what I mean? Like...for Term 1 (pointing gesture to Term 1) you will add like (making another gesture, see Figure 4).

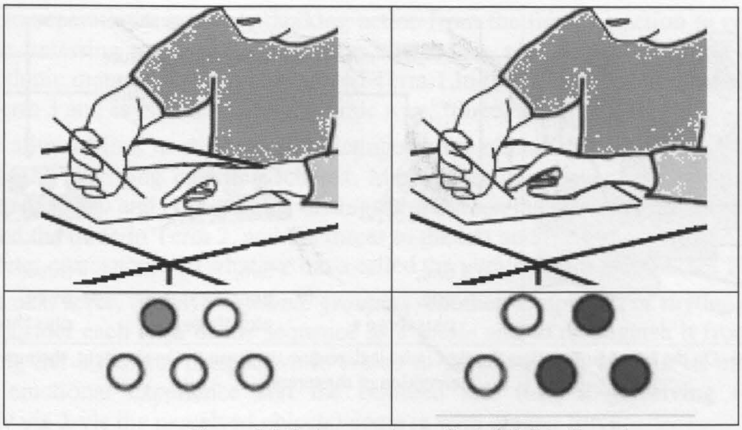

Figure 5. Mimi attempting to find digit 3 in Term 1

To explore the role that digit 3 may play, in line 9 Mimi refers to Term 1 and makes two indexical gestures. The first gesture indicates the first circle on the top of the first row (see Figure 5, bottom left). The second indexical gesture discloses a meaningful geometric-numeric link between digit 3 and the three sought-after circles in the term (see Figure 5, bottom right).

Mimi has not mentioned or pointed to the first circle on the bottom row. However, the circle has been noticed. Indeed, right after finishing her previous utterance, Mimi emphatically says "OK!" and proceeds to take into account all circles. She says:

10. Mimi: OK! It would be like one (indexical gesture on Term 1), one (indexical gesture on Term 1), plus three (grouping gesture); this (making the same set of gestures but now on Term 2) would be two, two, plus three; this (making the same set of gestures but now on Term 3) would be three, three, plus three.

Making two indexical gestures and one "grouping gesture" that surrounds the three last circles on Term 1, Mimi renders a specific configuration apparent to herself and to her group-mates. This set of three gestures is repeated as she moves to Term 2 and Term 3. The gestures are accompanied by the same sentence structure (see Figure 6). Through a rhythmic coordination of gestures and words, Mimi thereby notices a general structure in a dynamic way and moves from particular terms towards a grasping of the general term of the sequence.

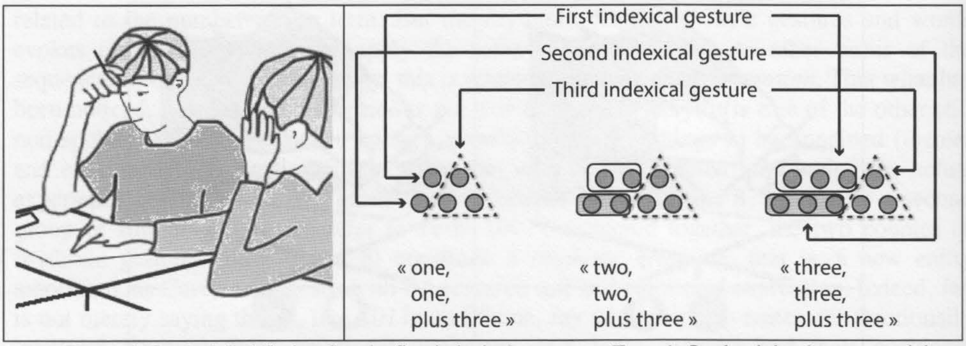

Figure 6. On the left, Mimi makes the first indexical gesture on Term 1. On the right, the new spatial perception of the terms

A prosodic analysis of Mimi's utterance conducted with the voice analysis software *Praat* (Boersma and Weenink 2006) revealed the temporal distribution of words and word intensity. In the top part of Figure 7, the waveform shows a visual distribution of words in time; the curve on the bottom shows the intensity of uttered words (measured in dB).

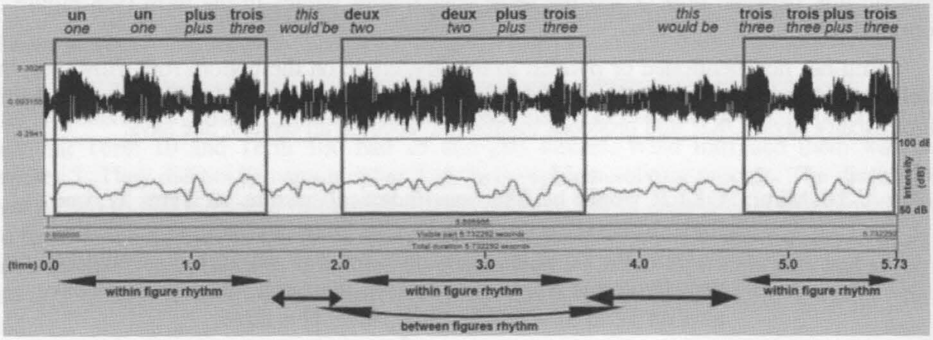

Figure 7. Prosodic analysis of Mimi's utterance conducted with Praat (from Radford, Bardini, and Sabena 2007: 522)

In addition to showing the temporal distribution of words, the prosodic analysis also gives us an idea of the various components of rhythm embedded in Mimi's multimodal intervention. Let us come back to Mimi's intervention.

In the first part (which corresponds to the utterance "one, one, plus three"), Mimi points three times, making sure that the pointing gestures are coordinated with words. The terms "one" and "one" are emphasized together and kept separated from the right after uttered term "three," accompanied by a grouping gesture. The operational term "plus" plays a two-fold role: it indicates that an addition has to be performed, but also that we are dealing with two separate things: ones on one side and the sought-after number three on the other. Then there is a pause to separate the previous thinking action from the thinking action to come. In the second part, referring to Term 2, Mimi says, "two, two, plus three." And she says it in a similar rhythmic manner as when she tackled Term 1.In the last part of her intervention, she refers to Term 3 and says in the same rhythmic way, "three, three, plus three."

Meter allows Mimi to rhythmically distinguish the circles that vary from those that do not vary in the emerging counting schema. Meter is the component of rhythm that helps Mimi to perceptually and conceptually distinguish between the ones and the three in Term 1, the twos and the three in Term 2, and the threes to the left and the three circles to the right in Term 3. Meter corresponds to what we have called the *within figure rhythm* (see Figure 7).

At the next level, we have *rhythmic grouping*, another component of rhythm that allows Mimi to consider each term of the sequence as a *whole* and to distinguish it from the other terms giving her a general perspective in which to perceive each term in its entirety. It is here that emotional experience and the centered self (i.e., a perceiving self-defined positionally vis-à-vis the perceived objects) come in with greater force.

There are still two important components of rhythm to be considered: *theme* and *prolongation*. As mentioned previously, *theme* is the very important component of rhythm that moves us from memory to imagination and that provides us with the feeling of continuity of the phenomenon under scrutiny—the sense that something will happen next, or the expectation of a forthcoming event (You 1994). The sense of a theme results from the rhythmic manner in which Mimi counts and moves from one term to the next. Mimi does not count the terms in a random manner, starting for instance by Term 3, then Term 1, then Term 2. There is a rhythm and an *order* to the counting that suggests that the process can continue further. Prolongation is the component of rhythm where a phenomenon is *expressed*. It happens when Mimi moves from Term 1 to 2, and when she moves from Term 2 to 3. Theme and prolongation may come together, as it happened in Mimi's group, when they could *imagine* Term 75 and still "see" and talk about the three circles there (Radford, Bardini, Sabena 2007).

We should bear in mind, though, that the various components of rhythm do not appear in a linear or sequential form, as the previous analysis may unfortunately intimate. As an organizer of thinking, rhythm brings together its various components at different moments with varied and evolving relations. Meter is a basic component deeply related to sensation and perception that provides a substrate to imagine and express things. Imagination and expression are the components of rhythm related to theme and expression, respectively. But in order to imagine and express something, different previously expressed elements need to come into relationship (rhythmic grouping). The picture that results from here is rather one of an evolving system whose elements mediate each other: thinking moves dialectically through the mediation of the components of rhythm, where, for instance, theme becomes or transforms itself into expression or expression becomes theme. As a concrete example, let us come back to Mimi's thinking. I argued that meter allowed Mimi to distinguish ones from three in Term 1. A theme emerges on the horizon; still the theme is only imagination. Through meter, the theme becomes prolongation when Mimi materially goes from Term 1 to Term 2. The previous theme reappears now re-invigorated and, like the bud in Hegel's example that transformed itself into a blossom, transforms itself into a new prolongation. We now have two terms of the sequence with a third term on the horizon. Mimi's thinking continues unfolding in the materialization of cultural mathematical thought and we now have not two terms but three. There is room now for putting into relationship the terms of the sequence between themselves. A new rhythmic entity emerges. From meter, theme, and prolongation appears *rhythmic grouping*—what we have called *between figures* rhythm (see Figure 7), that is, the appearance of a *regularity* between figures. The mode of emergence of rhythmic grouping should not be thought of as something occurring after the occurrence of something else. It emerges before, so to speak, or perhaps more accurately, *simultaneously* with the other components of rhythm.

Synthesis and concluding remarks

The argument I set out to support in this chapter was that mathematical thinking is something akin to poetry or drama: a phenomenon happening in time. More precisely, I argued that mathematical thinking is movement, the movement of thought. To elaborate my argument I suggested conceptualizing thought and thinking along the lines of two central Aristotelian ontological categories: *potentiality* (*δύναμις, dunamis*) and *actuality* (*ἐνέϱγεια, energia*). Thought, I argued, can be conceived of as potentiality, while thinking can be conceptualized as actuality. In other words, what I argued was that thinking is the actualization or materialization of thought.

The relationship between potentiality and actuality that I have suggested in this article is not a mere formal or correlative relation, but a *dialectical* one. I elaborated my point against the background of Hegel's concept of mediation, a concept underpinned by an idea of material and ideational worlds as something interrelated and in continuous motion and transformation. I argued that the mediation of thought is human, sensuous, practical, material, activity. Thinking appears hence as a phenomenon that unfolds in time, not as a pure intellectual or mental phenomenon, but as a fully material, corporeal, embodied, and sensuous phenomenon. Rhythm—as that which characterizes the reappearance of something that alternates or oscillates between symmetry and asymmetry, and between one and otherness—reveals itself as a central organizing principle of thinking. Drawing on Cureton's (2014) work, I considered four components of rhythm: meter, rhythmic grouping, theme, and prolongation. I referred to a Grade 9 mathematics lesson on algebraic generalization to show how the components of rhythm interact dialectically with each other. The analysis suggests that through indexical and grouping gestures, Mimi produced meter to emphasize some circles in the visual realm. However, pointing (and gestural activity in general) is not enough. Pointing quickly becomes ambiguous (Bühler 1979). Mimi also had recourse to words (e.g., "two," "three," "plus") with which the content of the gestures were disambiguated and endowed with theoretical content. The ensuing aural meaning of words was hence synchronized with kinesthetic and visual meanings encompassing the pointed circles and the successive position of gestures in the space. As the visual and kinesthetic meanings appeared, they gave rise to a dialectical dynamic unity of theme, prolongation, rhythmic grouping, and re-invigorated meter. This continuously evolving dialectical dynamic unity accounted for a subtle semiotic coordination that produced at the aural, kinesthetic, and visual levels a *regularity* that proved to be crucial for grasping a sensuous meaning of mathematical generality.

As our analysis implies, rhythm is not merely part of the pragmatic dimension of language and communication. Rather, rhythm—and this is the argument put forward in this article—is an integral part of mathematical thinking. What this means is that, even when we work alone on a mathematical problem, rhythm is present. The way we perceive signs (formulas, diagrams, tables), the way we move from signs to signs, and emphasize and de-emphasize things, is not, I would argue, very different from what the Grade 9 students featured in this article did. How these components of rhythm evolve and become "internalized" (to use Vygotsky's [1987] expression) is, of course, a matter of future research. Maybe these components of rhythm become contracted, like inner speech. While we wait to accumulate more evidence, let me finish by mentioning an episode that does not have anything to do with mathematics. It is an episode that has to do with chess. It happened in the chess championship played in Lyon in 1990 between Anatoly Karpov and Garry Kasparov. Kasparov has to make a crucial decision: to sacrifice or not to sacrifice his Queen. He is not thinking aloud, of course. Yet, at a certain moment, he again checks his calculations and, while quickly moving his eyes, makes a very fast, pointing gesture and goes through the possible moves that may follow the sacrifice of his Queen. Meter and perceptual pattern recognition produce, in the terminology that we have been using, theme and prolongation. The reconstruction of the last part of the game with a journalist allows us to see in more detail these aspects of rhythm in Kasparov's thinking. The episode can be found by clicking https://www.youtube.com/watch?v=SMe-hvCwTRo. The episode to which I am referring starts at 6:10. The fast, pointing gesture appears at 6:46. The reconstruction of the final part of the game goes from 0 to 6:09 minutes. At a certain point of the reconstruction (3:43), the journalist asks Kasparov if, when making the decision about the critical Queen sacrifice, he had more confidence in his calculations or in his feelings. Kasparov responds, "maybe in my feelings."

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